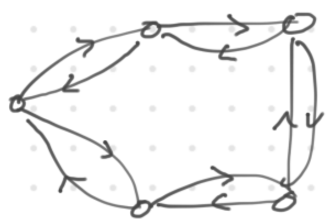
Flows

Def Directed graph G w/ capacities

$c: E(G) \to \mathbb{R}^+$ & we assume

$\forall$ edges (u,v) $\exists$ an edge (v,u)



A **flow** on G with source s & sink t is a function

$f: E(G) \to \mathbb{R}$ respecting the

1) $f(u,v) = -f(v,u)$ $\forall$ u,v
2) $f(u,v) \leq c(u,v)$
3) $\forall$ $v \neq s,t$

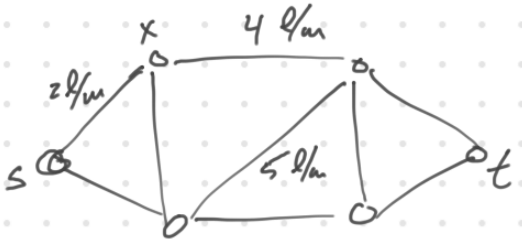
$$\sum_{\substack{(u,v): \\ (u,v) \in E}} f(u,v) = 0$$

we say:

1) - flow is symmetric
2) - respects the capacities
3) - has conservation of flow

The **value** of the flow is

$$\sum_{(s,u) \in E(G)} f(s,u)$$



Think of an undirected graph w/ edges corresponding to pipes w/ a capacity liters/min

we want to push water through the system while respecting the capacities on the edges

we could have flow from $s \to x$ or vice versa from $x \to s$

between s & x, we can have flow in both directions

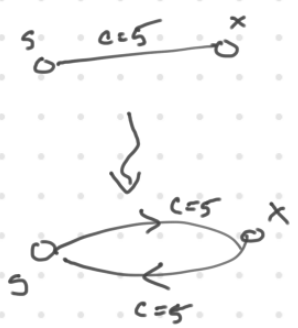
so a (+) flow from $s \to x$ is equivalent to a (−) flow in the opposite direction

This is the motivation for introducing the bidirectional edges in each direction & requirement that

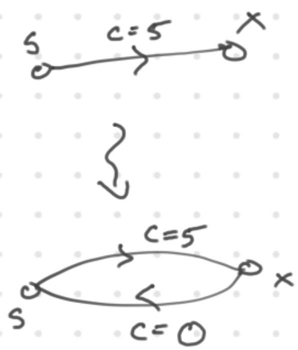
$$f(u,v) = -f(v,u)$$

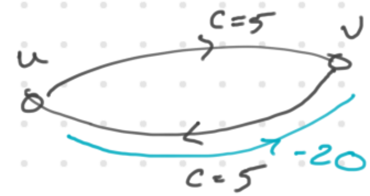
We could certainly imagine a situation where flow can only go in one direction eg ∃ values on the pipes which only allow flow from $s \to x$

undirected case



directed case





-20 on (v,u)

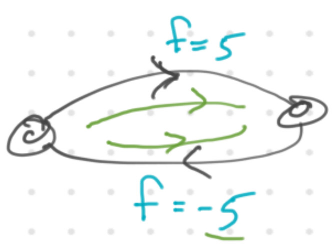
ie $f(v,u) = -20$

respects the capacity constraint on (v,u)

BUT since $f(u,v) = -f(v,u)$

$\Rightarrow f(u,v) = 20$ & violates the capacity of (u,v)

f is a flow, for each pair of vertices u, v connected (by double) edges



f has magnitude & sign

sign: direction

magnitude: amt of stuff going through pipe

Conservation of flow:

undirected model – ~~flow~~ water flowing into x is equal to flow out of x

directed model

flow into x is

Σ of positive values on edges into x $=$

flow out is

$=$ Σ of positive values on edges leaving x

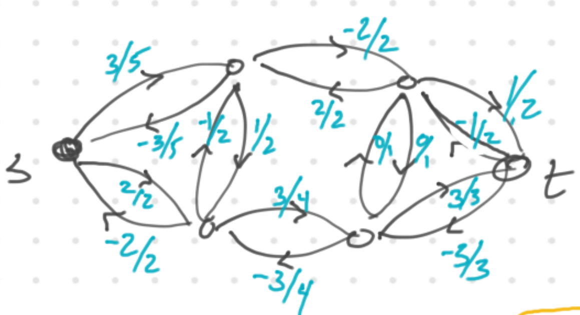
|| by symmetry

Σ of negative values on edges into x

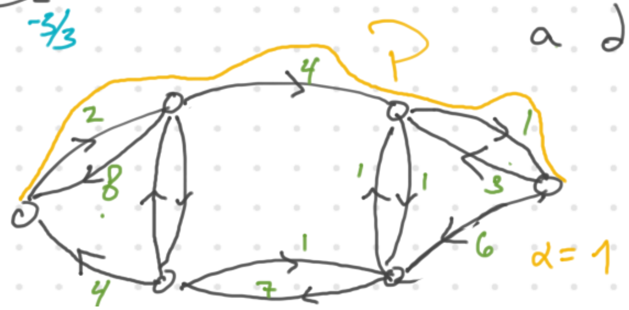
ie. $\sum_{(y,x)} f(y,x) = 0$ which is condition 3)

Def The residual graph is the directed graph G' formed by edges of G st $f(u,v) < c(u,v)$. We def weight function $r(u,v)$ for edges in G' w/ $r(u,v) = c(u,v) - f(u,v)$

ie "residual" amount of capacity on the edge (u,v)



residual graph



Note

1) $r(u,v)$ is always positive because $f(u,v) < c(u,v)$ so $c(u,v) - f(u,v)$ is (+) (because $c(u,v)$ is always ≥ 0)

2) $r(u,v) + r(v,u) =$
$= c(u,v) + c(v,u)$

Assume for the moment, $\exists$ a directed path from $s \to t$ in G' – call it P & let α be the $\min\limits_{(u,v)\in E(P)} r(u,v)$

Prop define

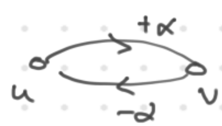
$$f'(u,v) := \begin{cases} f(u,v) & \text{if neither } (u,v) \text{ nor } (v,u) \in E(P) \\ f(u,v)+\alpha & \text{if } (u,v) \in E(P) \\ f(u,v)-\alpha & \text{if } (v,u) \in E(P) \end{cases}$$

Then f' is a flow of value = value of $f + \alpha$

pf

we need to prove 3 properties

1) is easy



either both $f'(u,v)$ & $f'(v,u)$ are unchanged, or we add α to one & subtract from the other

$\Longrightarrow \quad f'(u,v) = -f'(v,u)$

2) for any edge (u,v) where $f'(u,v)$ is increased

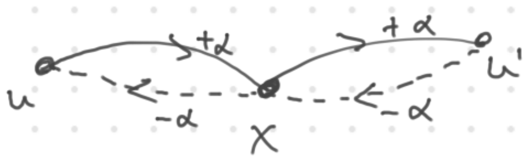
$f'(u,v) = f(u,v) + \alpha$ &

by choice of α,

$\alpha \leq c(u,v) - f(u,v)$.

$$\Rightarrow f'(u,v) \leq f(u,v) + (c(u,v) - f(u,v)) \leq c(u,v)$$

3) look at a vertex x ($\neq s$ or t). if $x \notin V(P)$, $f'(x,u) = f(x,u)$ $\forall u$ & so $\sum f'(x,u) = \sum f(x,u) = 0$ ✓



if $x \in V(P)$

if $x \in V(P)$, Then we have two edges $(u,x), (x,u') \in E(P)$ &

$$f'(u,x) = f(u,x) + \alpha$$
$$f'(x,u') = f(x,u') + \alpha$$

if we look at $\sum_{(v,x)\in E(G)} f'(v,x)$

we add α to one such edge, subtract α from one such edge & o.w. f' is unchanged

so overall, The sum is still 0.

Finally, The value increased by <u>exactly</u> α ∎

Alg (Ford - Fulkerson)

given a network G w/ capacities c

source s + sink t

set $f(u,v) = 0 \quad \forall (u,v) \in E(G)$

While $\exists$ a directed path P from

$s \to t$ in the residual graph G' Do

$\alpha := \min\limits_{(u,v)\in P} c(u,v) - f(u,v)$

$f(u,v) = f(u,v) + \alpha$ for all $(u,v) \in E(P)$

$f(u,v) = f(u,v) - \alpha$ for all $(v,u) \in E(P)$

return flow f.

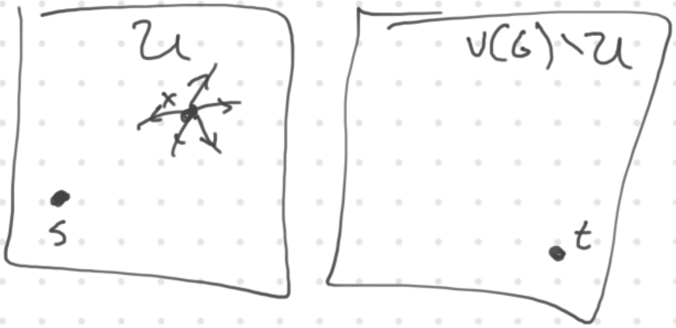
<u>Obs</u> if $c(u,v)$ is integer $\forall (u,v) \in E(G)$

$\Rightarrow$ algorithm above will terminate.

Since $c(u,v)$ is integer, f will always be an integer value + value of flow is strictly increasing with each pass of the while loop. Moreover, value of flow is always $\leq \sum\limits_{(s,x)\in E} c(s,x)$ always finite

$\Longrightarrow$ The algorithm must terminate.

Consider $\mathcal{U} \subseteq V(G)$ w/ $s \in \mathcal{U}$, $t \notin \mathcal{U}$



Consider $\displaystyle\sum_{x \in \mathcal{U}} \sum_{(x,y) \in E(G)} f(x,y)$ = value of f

$$= \sum_{\substack{(x,y) \in E(G) \\ \text{st } x \in \mathcal{U}}} f(x,y)$$

$$= \underbrace{\sum_{\substack{(x,y) \in E(G) \\ x,y \in \mathcal{U}}} f(x,y)}_{=0} + \sum_{\substack{(x,y) \in E(G) \\ x \in \mathcal{U},\, y \notin \mathcal{U}}} f(x,y) = \sum_{\substack{(x,y) \in E(G) \\ x \in \mathcal{U},\, y \notin \mathcal{U}}} f(x,y) \le \sum_{\substack{(x,y) \in E(G) \\ x \in \mathcal{U},\, y \notin \mathcal{U}}} c(x,y)$$

define this set of edges as the cut defined by $\mathcal{U}$.

define this to be the capacity of the cut defined by $\mathcal{U}$

for every s-t flow + cut defined by $\mathcal{U}$

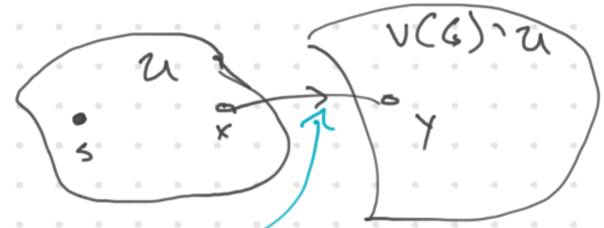
value of the flow $\leq$ capacity of the cut defined by $\mathcal{U}$

$\Rightarrow$ max value of an s-t flow $\leq$ min cap. of an s-t cut defined by $\mathcal{U}$.

return to Ford-Fulkerson, + let f be the flow returned by the algorithm.

if we consider the residual graph G' of f, then $\nexists$ a directed s-t path in G'

define $\mathcal{U} :=$ vertices reachable from s by a directed path in G'



$s \in \mathcal{U}$ trivially

$t \notin \mathcal{U}$ because $\nexists$ an s-t path in G'

moreover, for every edge (x,y) of G w/ $x \in \mathcal{U}$, $y \notin \mathcal{U}$ such an edge is not present in G' $\Rightarrow$ $f(x,y) = c(x,y)$

$\Rightarrow$

$$\text{value of } f = \sum_{\substack{(x,y)\in E(G) \\ x\in \mathcal{U},\, y\notin \mathcal{U}}} f(x,y) = \sum_{\substack{(x,y)\in E(G) \\ x\in \mathcal{U},\, y\notin \mathcal{U}}} c(x,y)$$

we already showed

$$= \text{Capacity of That cut}$$

<u>Conclusion</u> f is an optimal flow & FF returns an optimal flow.

in The sense it has maximum value.

<u>Conclusion</u> <u>Thm</u> max value of a flow = min capacity of an s-t cut